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ON THE DETERMINATION OF A CATENARY WITH GIVEN DIRECTRIX AND PASSING THROUGH TWO GIVEN POINTS

BY HARRIS F. MACNEISH

1. The problem of determining a catenary with given directrix and passing through two given points, which plays an important part in the problem of the surface of revolution of minimum area, has been studied by :—

F. N. Goldschmidt, *Göttingen Prize Essay*, 1831 ;

L. L. Lindelöf-Moigno, *Calcul des variations*, 1861, no. 103 ;

J. Dienger, *Grundriss der Variationsrechnung*, 1867, pp. 15–19 ;

I. Todhunter, *Researches in the Calculus of Variations*, 1871, pp. 60–67 ;

H. A. Schwarz, *Lectures on the Calculus of Variations*, Summer 1892 (unpublished) ; for Schwarz's solution see

Harris Hancock, "On the number of catenaries through two points," ANNALS OF MATHEMATICS, ser. 1, vol. 10 (1896), pp. 159–174, and *Lectures on the Calculus of Variations*, 1904, chapter III.

If we take the given directrix for the x-axis and denote the coordinates of the two given points A and B by (x_0, y_0) and (x_1, y_1) respectively, the required catenary may be written : *

$$y = m \operatorname{ch} \frac{x - x_0'}{m}, \tag{1}$$

where x_0', m are the constants of integration to be determined by the initial conditions expressed by the two equations

$$y_0 = m \operatorname{ch} \frac{x_0 - x_0'}{m}, \tag{2}$$

and

$$y_1 = m \operatorname{ch} \frac{x_1 - x_0'}{m}. \tag{3}$$

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* We use Laisant's notation *ch*, *sh*, and *coth*, for hyperbolic cosine, hyperbolic sine, and hyperbolic cotangent.

The problem is evidently possible only if A and B are on the same side of the x-axis; we therefore suppose $y_0 > 0$, $y_1 > 0$. We may further suppose without loss of generality that $x_0 < x_1$.

The following is the result given by Schwarz:

There exists one and but one quantity μ which satisfies the equation *

$$\phi(\mu) = \frac{y_1\mu}{\sqrt{y_1^2 - \mu^2}} + \frac{y_0\mu}{\sqrt{y_0^2 - \mu^2}} - (x_1 - x_0) = 0, \tag{4}$$

and which lies between the values 0 and the smaller of the two quantities y_0 and y_1.† If now we form with this quantity μ the expression

$$f_2(\mu) = \log\left(\frac{y_1 + \sqrt{y_1^2 - \mu^2}}{\mu}\right) + \log\left(\frac{y_0 + \sqrt{y_0^2 - \mu^2}}{\mu}\right) - \frac{x_1 - x_0}{\mu},$$

we have the following three cases:

(I) $f_2(\mu) < 0$: *no real catenary;*
(II) $f_2(\mu) = 0$: *one real catenary;*
(III) $f_2(\mu) > 0$: *two real catenaries.*

Moreover in Case II μ is the value of m for the unique catenary through the points A and B.

2. We now consider one of the two given points, say A, as fixed and propose to determine the curve upon which the point B must lie in order that there shall be but one catenary.

If the point A is fixed, it is evident that the curve upon which B must lie can be determined from equations (4) and $f_2(\mu) = 0$ by eliminating μ and considering x_1, y_1 as the running coordinates of the curve.

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* All square root signs are to be taken as positive.

† Schwarz's results are obtained on the assumption that $y_1 \geqq y_0$. By the substitution

$$\bar{x}_0 = -x_1; \quad \bar{x}_1 = -x_0$$
$$\bar{y}_0 = y_1; \quad \bar{y}_1 = y_0$$
$$\bar{m} = m; \quad \bar{x}_0' = -x'$$

the results can be shown to hold for $y_1 < y_0$.

The curve can, however, be obtained more easily by introducing two quantities u_0 and u_1 * determined by the relations

$$\begin{aligned} y_0 &= \mu \operatorname{ch} u_0, \\ y_1 &= \mu \operatorname{ch} u_1, \\ u_0 &< 0, \quad u_1 > 0. \end{aligned} \tag{5}$$

These conditions determine u_0 and u_1 uniquely.

From equations (4) and (5),

$$\frac{\mu \operatorname{ch} u_1}{\operatorname{sh} u_1} - \frac{\mu \operatorname{ch} u_0}{\operatorname{sh} u_0} - (x_1 - x_0) = 0. \tag{7}$$

With regard to the sign of the second term notice that $\operatorname{sh} u_0 = -(\sqrt{y_0^2 - \mu^2})/\mu$, since u_0 and therefore $\operatorname{sh} u_0$ is negative.

Equation (7) reduces to

$$\coth u_1 - \coth u_0 = \frac{x_1 - x_0}{\mu}, \tag{8}$$

and the equation $f_2(\mu) = 0$ becomes

$$\log(\operatorname{ch} u_1 + \operatorname{sh} u_1) + \log(\operatorname{ch} u_0 - \operatorname{sh} u_0) = \frac{x_1 - x_0}{\mu},$$

or

$$u_1 - u_0 = \frac{x_1 - x_0}{\mu}. \tag{9}$$

Now from (5), $\mu = y_0/\operatorname{ch} u_0$, whence from (9),

$$x_1 = x_0 + \frac{y_0(u_1 - u_0)}{\operatorname{ch} u_0}. \tag{10}$$

Also from (5),

$$y_1 = y_0 \frac{\operatorname{ch} u_1}{\operatorname{ch} u_0}, \tag{11}$$

and from (8) and (9),†

$$\coth u_0 - u_0 = \coth u_1 - u_1. \tag{12}$$

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* The quantities u_0 and u_1 have a simple geometric meaning. By reference to equations (2) and (3) it is seen that $u_0 = \frac{x_0 - x_0'}{\mu}$ and $u_1 = \frac{x_1 - x_0'}{\mu}$ for the unique catenary passing through the points A and B.

† Since equation (12) is the same as the equation for the determination of the conjugate points (see Bolza, *Calculus of Variations*, p. 64), our curve is identical with the envelope of the set of catenaries through A.

These equations (10), (11), (12) give us the required curve in parameter representation.

3. To determine the graph from these equations, we may choose either u_0 or u_1 as the independent parameter, but it is more symmetrical to introduce a third parameter t determined by the relation

$$\coth u_0 - u_0 = \coth u_1 - u_1 = t. \tag{13}$$

Then corresponding values of u_1 and u_0 are obtained from the two branches of the graph of the equation

$$t = \coth u - u,$$

since from (5) $u_0 < 0$ and $u_1 > 0$.

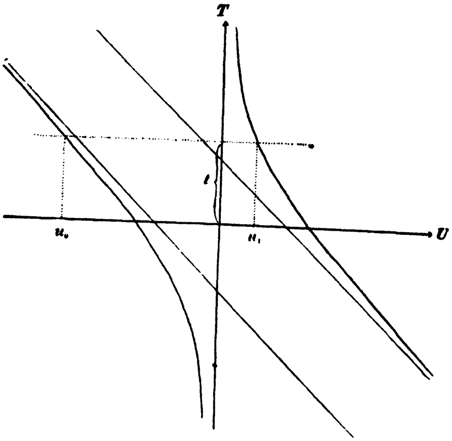

FIG. 1.

From the left branch are obtained the values of u_0, from the right branch the corresponding values of u_1, for a given value of t.

Let us now consider the derivatives $\frac{dx_1}{dt}$ and $\frac{dy_1}{dt}$.

From (13)
$$\frac{dt}{du_1} = -\coth^2 u_1,$$

and
$$\frac{dt}{du_0} = -\coth^2 u_0;$$

then from (10) and (11)

$$\frac{dx_1}{dt} = y_0 \frac{\operatorname{sh}^3 u_0 \operatorname{ch}^3 u_1 - \operatorname{sh}^3 u_1 \operatorname{ch}^3 u_0}{\operatorname{sh} u_1 \operatorname{ch}^2 u_1 \operatorname{ch}^4 u_0},$$

and

$$\frac{dy_1}{dt} = y_0 \frac{\operatorname{sh}^3 u_0 \operatorname{ch}^3 u_1 - \operatorname{sh}^3 u_1 \operatorname{ch}^3 u_0}{\operatorname{ch}^2 u_1 \operatorname{ch}^4 u_0}.$$

Since

$$u_0 < 0 \quad \text{and} \quad u_1 > 0,$$

$$\frac{dx_1}{dt} < 0 \quad \text{and} \quad \frac{dy_1}{dt} < 0,$$

and as t goes from $-\infty$ to $+\infty$, x_1 goes from ∞ to x_0 and y_1 goes from ∞ to 0; we can therefore introduce x_1 as independent variable.

Then*

$$\frac{dy_1}{dx_1} = \operatorname{sh} u_1,$$

so that the slope of the curve is always positive, since u_1 is positive.

When $\frac{dy_1}{dx_1} = 0$, $u_1 = 0$, and hence $u_0 = -\infty$, from equation (13).

Now

$$\frac{d^2 y_1}{dx_1^2} = \operatorname{ch} u_1 \frac{du_1}{dx_1} = \operatorname{ch} u_1 \frac{du_1}{dx_1} \cdot \frac{dt}{dx}$$

$$= \frac{\operatorname{ch} u_1 \operatorname{sh}^3 u_1 \operatorname{ch}^4 u_0}{\operatorname{sh}^3 u_1 \operatorname{ch}^3 u_0 - \operatorname{sh}^3 u_0 \operatorname{ch}^3 u_1};$$

hence $\frac{d^2 y_1}{dx_1^2}$ is positive, since $\operatorname{sh} u_1$, $\operatorname{ch} u_1$, and $\operatorname{ch} u_0$ are always positive and $\operatorname{sh} u_0$ is always negative. Therefore the curve is convex towards the x-axis. Now in equations (10), (11), (12) we can always make a similarity transformation to make $x_0 = 0$, $y_0 = 1$. For these values of x_0, y_0 the following table has been computed:

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* This is the same as the slope of the tangent to the catenary, which verifies the statement that our curve is the envelope of the set of catenaries.

u_1	t	u_0	x_1	y_1
0.00	∞	$-\infty$	0.00	0.00
0.25	3.83	–4.84	0.08	0.02
0.50	1.66	–2.67	0.45	0.16
0.75	0.82	–1.87	0.79	0.39
1.00	0.31	–1.42	1.11	0.70
1.20	0.00	–1.20	1.33	1.00
1.50	–0.39	–0.93	1.66	1.61
2.00	–0.96	–0.69	2.16	3.02
2.50	–1.49	–0.53	2.65	5.36
3.00	–1.99	–0.43	3.14	9.20
3.50	–2.50	–0.35	3.63	15.61
4.00	–3.00	–0.30	4.11	26.13
∞	$-\infty$	0.00	∞	∞

We then obtain the graph shown in Figure 2. The symmetrical branch to the left of $x = x_0$ corresponds to the case when $x_1 < x_0$.

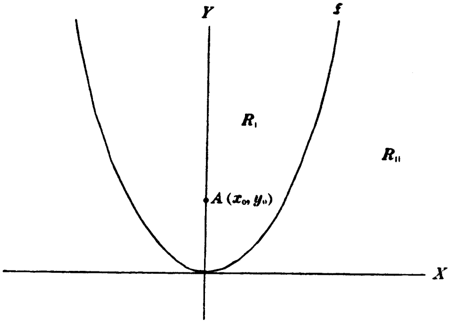

FIG. 2.

As has been shown in §2 our curve can be represented in the form

$$y = \phi(x);$$

and as x increases from $-\infty$ to $+\infty$, y decreases from $+\infty$ to 0 and then increases from 0 to $+\infty$. Hence our curve divides the first quadrant into two regions, R_{I} and R_{II}, the first above, the second below the curve. In R_{I}, $y > \phi(x)$; and in R_{II}, $y < \phi(x)$. We shall prove that $f_2(\mu) > 0$ throughout R_{I}, and that $f_2(\mu) < 0$ throughout R_{II}.

For, from the theorem on implicit functions, it follows that the implicit function $\mu = \mu(x_1, y_1)$ defined by equation (4) is continuous in the first quadrant, and therefore the function $F(x_1, y_1) = f_2[\mu(x_1, y_1)]$ is also continuous. But $F(x_1, y_1) = 0$ is by definition the equation of the curve (10), (11). Hence $F(x_1, y_1)$ must have a constant sign throughout R_{I}, and a constant sign throughout R_{II}. In order to determine these signs, it is therefore sufficient to determine them for one point of R_{I} and for one point ot R_{II}. For this purpose we give y_1 the value $y_1 = y_0$; equation (4) then reduces to

$$\frac{2y_0\mu}{\sqrt{y_0^2 - \mu^2}} = x_1 - x_0,$$

or

$$\mu = \frac{(x_1 - x_0)y_0}{\sqrt{(x_1 - x_0)^2 + 4y_0^2}}.$$

Hence

$$\lim_{x_1 = x_0} \mu = 0 \quad \text{and} \quad \lim_{x_1 = x_0} \frac{x_1 - x_0}{\mu} = 2.$$

Therefore

$$\lim_{x_1 = x_0} f_2(\mu) = +\infty,$$

so that $f_2(\mu) > 0$ for the point $x_1 = x_0$, $y_1 = y_0$ of the region R_{I}.

Again,

$$\lim_{x_1 = \infty} \mu = y_0.$$

Therefore

$$\lim_{x_1 = \infty} f_2(\mu) = -\infty,$$

so that $f_2(\mu) < 0$ for the point $x_1 = \infty$, $y_1 = y_0$ of the region R_{II}.

This proves the above statement, and we obtain therefore the result:

1) *From the point A to any point of R_{I} two catenaries can be drawn with the x-axis as directrix.*

2) *From the point A to any point of the curve $\mathfrak{F}$ only one catenary can be drawn with the x-axis as directrix.*

3) *From the point A to any point of the region R_{II} no catenary can be drawn with the x-axis as directrix.*